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Feynman Rules for Probability Amplitudes

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Starting with very simple assumptions, Feynman rules for the quantum mechanical amplitudes and the associated probabilities are derived. These rules emerge as the only consistent rules for manipulating complex amplitudes assigned to processes. The probability of a process to which an amplitude x has been assigned is determined as $p(x)=|x|^\alpha$, $0<\alpha\leq 2$. If virtual processes are allowed, $\alpha=2$.

In the Feynman approach to quantum mechanics the basic entities are processes or transitions rather than states (Feynman, 1985; Feynman *et al.*, 1965; Feynman and Hibbs, 1965). A process is an *ordered* event starting at a given initial state, proceeding possibly via intermediate states, and ending up at a definite final state. To each possible process one assigns a complex number called the probability amplitude for the process. The assigned number is assumed to depend only on the given process and to be independent of the past history (Markovian property).

One of the great merits of Feynman's formulation of quantum mechanics is the clear separation between kinematics and dynamics. The former sets the stage and gives the rules of the game, namely, rules for *combining* amplitudes and for calculating the associated probabilities. The latter selects the particular dynamical model for *assigning* amplitudes (e.g., nonrelativistic or relativistic quantum dynamics). The situation is analogous to that existing in probability theory. The axioms of probability determine the general laws for combining probabilities. The user then selects a particular model for assigning probabilities appropriate for the given problem.

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That quantum kinematics alone has far-reaching consequences has been amply demonstrated, especially by Feynman [see, in particular, Feynman (1985) and Feynman *et al.* (1965)]. One can state the laws of quantum kinematics as follows: (1) The amplitude for processes occurring in succession is the product of the amplitudes for each part; (2) if a process can proceed in more than one way, the amplitude for the process is the sum of the amplitudes for the different alternatives, and (3) the probability for a process is the absolute value squared of the amplitude for the process. A vivid illustration of these laws is supplied by the familiar two-slit experiment (Feynman and Hibbs, 1965; Feynman *et al.*, 1965). Electrons emerging from a source S arrive at a detector placed behind a vertical screen through two horizontal parallel slits cut in the screen. Let x stand for the distance of the detector, measured along the vertical, from a point midway between the two slits. If $\langle x|S\rangle$ denotes the amplitude of the electron to go from S to x, then

$$\langle x|S\rangle = \langle x|S\rangle_{\text{via }1} + \langle x|S\rangle_{\text{via }2}$$

$$\langle x|S\rangle_{\text{via }i} = \langle x|i\rangle\langle i|S\rangle, \qquad i=1,2$$

and

$$p_{12} = p(\langle x|S\rangle) = p_1 + p_2 + \langle x|S\rangle_1^*\langle x|S\rangle_2 + \langle x|S\rangle_2^*\langle x|S\rangle_1$$

Here $\langle x|S\rangle_i$ is the amplitude to proceed from S to x via slit i and $p_i = |\langle x|S\rangle_i|^2$ is the probability for a process in which slit i alone is open. The interference term in the last expression exhibits the peculiarities of quantum behavior, which may appear to contradict our basic notions about probabilities (Feynman and Hibbs, 1965; Feynman *et al.*, 1965). Yet, it is an established *empirical* fact that nature does obey the laws of quantum kinematics. Is there any way to understand the "machinery" behind these laws? Could these laws be derived from simpler, intuitive assumptions? Are the laws consistent? How would the laws change if some of the basic assumptions were changed?

In what follows we shall apply a method pioneered by Cox (1946) in a different context[3] to *derive* the laws of quantum kinematics from very simple assumptions. These laws will emerge as the only consistent laws for combining amplitudes and for calculating the associated probabilities. Given a process with an amplitude x, we shall also establish the rule $\bar{x} = x^*$ for the amplitude of the inverse process.

Let A, B, $C, \ldots$ denote states and let $\langle AB|C\rangle$ denote the complex number associated with the process $C \to B \to A$ (Figure 1a). We shall assume

[3]The landmark paper by Cox (1946) establishes the rules for manipulating probabilities assigned to propositions or "events."

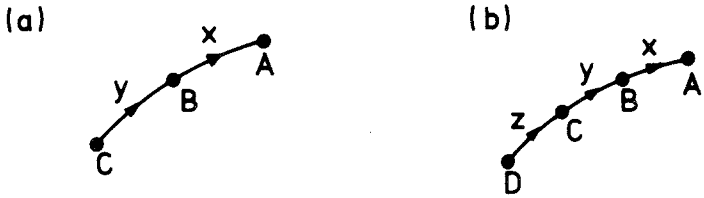

Fig. 1. Processes in series.

that the amplitude $\langle AB|C\rangle$ is a function of the two partial amplitudes $\langle B|C\rangle = y$ and $\langle A|B\rangle = x$:

$$\langle AB|C\rangle = f(x, y)$$

Consider the process $D \to C \to B \to A$ (Figure 1b). By the associative law for combining processes in series, the function $f(x, y)$ must satisfy

$$f[x, f(y, z)] = f[f(x, y), z] \tag{1}$$

Consider now processes in parallel (Figure 2a) and assume

$$\langle A|B\rangle = g(x, y)$$

where x and y are the amplitudes assigned to the two branches. Clearly, $g(x, y)$ is commutative,

$$g(x, y) = g(y, x) \tag{2}$$

and by the associative law for combining processes in parallel (Figure 2b),

$$g[x, g(y, z)] = g[g(x, y), z] \tag{3}$$

Let e denote the amplitude for the impossible process. (We assume that the amplitudes for any given process tend to one and the same limit e as the process becomes blocked.) Then the amplitude for the process shown in Figure 2a, with the lower branch blocked, must satisfy

$$g(x, e) = x \tag{4}$$

Fig. 2. Processes in parallel.

We now turn to the process displayed in Figure 3. Viewed as two processes in series Figure 3a), the amplitude for the process is given by $f[g(x, y), \lambda]$. On the other hand, viewed as a process in parallel (Figure 3b), the amplitude is $g[f(x, \lambda), f(y, \lambda)]$. Equating the two expressions, we obtain the distributive law

$$f[g(x, y), \lambda] = g[f(x, \lambda), f(y, \lambda)] \tag{5}$$

Equations (1)-(5) spell the properties that the functions $f(x, y)$ and $g(x, y)$ must have. It is evident that the choice

$$f(x, y) = xy, \qquad g(x, y) = x + y, \qquad e = 0 \tag{6}$$

satisfies all the requirements (1)-(5). What is perhaps less evident is that equations (1)-(5) together with the assumption of analyticity in both variables x and y single out the solution above as (essentially) the only solution to these equations (Tikochinsky, in press). That is, given $f(x, y)$ and $g(x, y)$ analytic in both variables and satisfying equations (1)-(5), there exists (at least locally) a one-to-one transformation $x' = H(x)$ such that both equations

$$H[f(x, y)] = H(x) \cdot H(y) \tag{7a}$$

and

$$H[g(x, y)] = H(x) + H(y) \tag{7b}$$

are satisfied. Thus, up to a form (but with an invariant content!) the amplitudes for processes in series and in parallel must obey the laws of quantum kinematics (6). Again, the situation is analogous to that prevailing in probability theory (Cox, 1946). One can choose if one likes to work with $\log p_i$ instead of p_i, thereby changing the form of the laws of probability without changing their content. Although analyticity is sufficient to establish the result (7), it is not clear whether it is necessary. That a strong demand must be put on the functions f and g in order to satisfy (7) is shown by the following example. Let $x = x_R + ix_I$. Then $g(x, y) = x + y$, $e = 0$, and $f(x, y) = x_R y_R + ix_I y_I$ satisfy equations (1)-(5) but not (7).

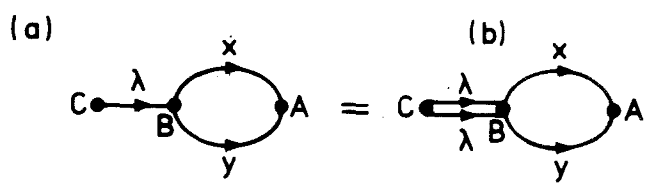

Fig. 3. Combined processes.

We turn now to determine the amplitude $\bar{x} = \langle B|A \rangle$ for the process $A \to B$ inverse to $B \to A$ (Figure 4a). Assuming a (1-1) continuous correspondence between the amplitudes x and $\bar{x}$, namely, $\bar{x} = s(x)$, $x = s(s(x))$, we have from Figures 1 and 2

$$\overline{xy} = \bar{x}\bar{y} \qquad \text{and} \qquad \overline{x+y} = \bar{x} + \bar{y} \tag{8}$$

and hence $\bar{1} = 1$, $\overline{x^{-1}} = (\bar{x})^{-1}$, $\bar{0} = 0$, $(\overline{-x}) = -\bar{x}$. Thus, all integers, rationals, and, by continuity, all real numbers x satisfy $\bar{x} = x$. We therefore have

$$\overline{x+x^*} = \bar{x} + \overline{x^*} = x + x^* \qquad \text{and} \qquad \overline{xx^*} = \bar{x}\overline{x^*} = xx^* \tag{9}$$

Solving these equations for the unknowns $\bar{x}$ and $\overline{x^*}$, we obtain for the inverse process

$$\bar{x} = x^* \qquad \text{or} \qquad \bar{x} = x \tag{10}$$

At this stage the solution $\bar{x} = x$ cannot be ruled out.

Finally, let us work out the connection between amplitudes and probabilities. Assuming that the probability for the process $B \to A$ is a function of the amplitude for that process, $p(A|B) = h(x)$, we have (Figure 1a)

$$h(xy) = h(x)h(y) \tag{11}$$

Hence $h(0) = 0$, $h(1) = 1$, and $h(x^{-1}) = h(x)^{-1}$. Since $h(x)^{-1} \geq 1$, the amplitudes x must be confined either to $|x| \leq 1$ or to $|x| \geq 1$. In order to include the impossible process with amplitude $x = 0$ and probability $p = h(0) = 0$, we choose the interior and boundary of the unit circle as the allowed domain for the amplitudes. On the unit circle, equation (11) gives $h(e^{i\theta} e^{-i\theta}) = h(e^{i\theta})h(e^{-i\theta}) = 1$. Hence $h(e^{i\theta}) = 1$ and $h(|x|\, e^{i\theta}) = h(|x|)h(e^{i\theta}) = h(|x|)$. Thus, the probability for a process is a function of the absolute value of the amplitude for the process satisfying $h(|x|\,|y|) = h(|x|)h(|y|)$. The general solution of the last equation, subject to $h(1) = 1$, is

$$p(A|B) = h(x) = |x|^\alpha, \qquad \alpha > 0 \tag{12a}$$

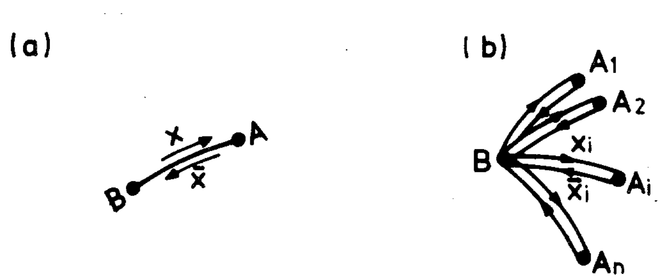

Fig. 4. Direct, inverse, and virtual processes.

In order to find out the range of possible α values, consider the process $B \to B$ starting at B, proceeding via all possible alternatives $B \to A_i \to B$ (Figure 4b), $i = 1, 2, \ldots, n$. Using equations (6) and (10) with $\bar{x} = x^*$, we obtain the amplitude for this process as $\langle B | B \rangle = \sum x_i^* x_i$. Since

$$p(B | B) = |\langle B | B \rangle|^\alpha = (\sum |x_i|^2)^\alpha \leq 1, \qquad \alpha > 0$$

we have $\sum |x_i|^2 \leq 1$. Now, by assumption, the set $A_1, \ldots, A_n$ is a mutually exclusive and exhaustive set of alternatives. Hence, $\sum |x_i|^2 \leq 1 = \sum |x_i|^\alpha$ and α satisfies

$$0 < \alpha \leq 2 \tag{12b}$$

This is as far as we can go in pinning down the probability associated with the amplitude x. [Note that by choosing $x \in [0, 1]$ and $\alpha = 1$, classical (Markovian) probability calculus is recovered.] In order to achieve further progress, we must make additonal assumptions about the nature of inverse processes. We shall assume that these processes proceed backward in time, and that combined virtual processes proceeding forward and then backward in time are possible processes, obeying the rules (6) and (10). The probability for the virtual process $B \to B$ starting at B at time $t = t_0$, proceeding via all possible alternatives $B \to A_i \to B$, and ending up at B at time $t_0 + 0$ is then $p(B | B) = 1 = \sum |x_i|^\alpha$, $(B | B)$ being the certain event. But $p = 1$ is obtained only for $|x| = 1$. Thus,

$$|\langle B | B \rangle| = 1 = \sum |x_i|^\alpha = \langle B | B \rangle = \sum |x_i|^2$$

and $\alpha = 2$. Note that the last relation is precisely the completeness relation $\langle B | B \rangle = \sum \langle B | A_i \rangle \langle A_i | B \rangle$. Incidentally, had we started with the other option for the amplitude of the inverse process, namely $\bar{x} = x$, we would have obtained $1 = |\sum x_i^2| = \sum |x_i|^\alpha$—a totally unacceptable result. (Consider, for example, $n = 2$ alternatives with $x_2 = ix_1$.) Thus, the second solution $\bar{x} = x$ in equation (10) must be discarded. To summarize, the probability of a process with an amplitude x is $p(x) = |x|^2$. Note that quantum kinematics could be established in the same way over the field of real numbers alone. The necessity of invoking complex amplitudes arises from quantum dynamics.

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